MATH 54 - HINTS TO HOMEWORK 7

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Here are a couple of hints to Homework 7! Enjoy! :)

Section 6.1: Inner products, lengths, and orthogonality

6.1.9. Divide the vector by its length!

6.1.21. What they mean by the transpose definition is $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$. Use the facts that $(A+B)^T = A^T + B^T$ and $(cA)^T = cA^T$.

6.1.21. Use the fact that $\|\mathbf{w}\|^2 = \mathbf{w} \cdot \mathbf{w}$ for any $\mathbf{w}$, and expand the left-hand-side out using a distributive law similar to $(a+b)(c+d) = ac + ad + bc + bd$.

Section 6.2: Orthogonal sets

Remember: A set $\mathcal{B}$ is orthogonal if for every pair of distinct vectors $\mathbf{u}$ and $\mathbf{v}$, $\mathbf{u} \cdot \mathbf{v} = 0$. It is orthonormal if it is orthogonal and every vector has length 1. An orthogonal set can be made orthonormal by dividing every vector by its length.

6.2.9.

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} \frac{\mathbf{x} \cdot \mathbf{u_1}}{\mathbf{u_1} \cdot \mathbf{u_1}} \\ \frac{\mathbf{x} \cdot \mathbf{u_2}}{\mathbf{u_2} \cdot \mathbf{u_2}} \\ \frac{\mathbf{x} \cdot \mathbf{u_3}}{\mathbf{u_3} \cdot \mathbf{u_3}} \end{bmatrix}$$

6.2.11, 6.2.15. The formula for orthogonal projection of $\mathbf{y}$ on the line spanned by $\mathbf{u}$ is:

$$\hat{\mathbf{y}} = \left(\frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \right) \mathbf{u}$$

The distance between $\mathbf{u}$ and L is then $\|\mathbf{y} - \hat{\mathbf{y}}\|$

Section 6.3: Orthogonal projection

Here are all the basic facts that you'll need:

(1) If $W = Span\{\mathbf{u_1}, \mathbf{u_2} \cdots \mathbf{u_k}\}$, then the orthogonal projection of $\mathbf{y}$ onto W is:

$$\hat{\mathbf{y}} = \left(\frac{\mathbf{y} \cdot \mathbf{u_1}}{\mathbf{u_1} \cdot \mathbf{u_1}} \right) \mathbf{u_1} + \left(\frac{\mathbf{y} \cdot \mathbf{u_2}}{\mathbf{u_2} \cdot \mathbf{u_2}} \right) \mathbf{u_2} + \cdots + \left(\frac{\mathbf{y} \cdot \mathbf{u_k}}{\mathbf{u_k} \cdot \mathbf{u_k}} \right) \mathbf{u_k}$$

(2) Then $\hat{\mathbf{y}}$ is in W, $y - \hat{\mathbf{y}}$ is in $W^{\perp}$ (that is, orthogonal to W).

(3) $\mathbf{y} = (\hat{\mathbf{y}}) + (y - \hat{\mathbf{y}})$, which decomposes $\mathbf{y}$ as a sum of two vectors, one in W and the other one orthogonal to W.

(4) $\hat{\mathbf{y}}$ is the closest point to $\mathbf{y}$ in W.

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(5) $\|y - \hat{\mathbf{y}}\|$ is the smallest distance between $\mathbf{y}$ and W.

6.3.21.

(a) **T**
(b) **T**
(c) **F**
(d) **T**
(e) **T**

SECTION 6.4: THE GRAM-SCHMIDT PROCESS

Use the formula given in Theorem 11. To get an *orthonormal* basis, just divide every vector at the end by its length. At every step, it's helpful to multiply your vector by a scalar to avoid fractions. This is ok, because you'll normalize them at the end anyway!

6.4.13. Here's the trick: If $A = QR$, then $Q^T A = Q^T QR = R$, since $Q^T Q = I$ (because the columns of Q are orthonormal). Hence:

$$R = Q^T A$$

SECTION 6.5: LEAST SQUARES PROBLEMS

Here's the general procedure to solve least-squares problems: To solve $A\mathbf{x} = \mathbf{b}$ in the least-squares sense, multiply both sides by A^T, and solve the (easier) equation $A^T A\hat{\mathbf{x}} = A^T\mathbf{b}$. Your solution $\hat{\mathbf{x}}$ is called the least-squares solution. The least squares error is $\|A\hat{\mathbf{x}} - \mathbf{b}\|$.

6.5.13. No! Because $\|A\mathbf{v} - \mathbf{b}\|$ is **smaller** than $\|A\mathbf{u} - \mathbf{b}\|$, so $\mathbf{u}$ **cannot** be a least-squares solution of $A\mathbf{x} = \mathbf{b}$, by the **definition** of a least-squares solution! (beginning of section 6.5)

6.5.15. Use the formula in theorem 15.

6.5.17.

(a) **T**
(b) **T**
(c) **F**
(d) **T**
(e) **T**